

SIMULTANEOUS EMBEDDINGS OF FINITE DIMENSIONAL DIVISION ALGEBRAS

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A celebrated theorem of P.M. Cohn [C] says that for any two division rings (not necessarily finite dimensional) over a field F , their amalgamated product over F is a domain which can be embedded in a division ring. Note that even with the two initial division rings being finite dimensional over their centers, the resulting division ring is **never** finite dimensional over its center. Perhaps this led Lance Small to ask the following question. We say D/F is a division algebra when D is a division ring finite dimensional over its center F . Assume F_1 and F_2 are fields with the same characteristic. Small asked whether any two division algebras D_1/F_1 and D_2/F_2 can be embedded in some third division algebra E/F .

We start with a surprisingly straightforward counterexample in the next section, but then show that a positive solution exists for division algebras finitely generated over a common subfield which is either algebraically closed or the prime subfield (Theorem 2.9).

1. A COUNTEREXAMPLE

Suppose, first of all, that D_1/F_1 is a f.d. division algebra embedded in the division algebra E/F , so F_1 and F share the same prime subfield P . There is a tower of subalgebras $F \subseteq F_1F \subseteq D_1F \subseteq E$, where F_1F must be an amalgamation of F_1 and F (meaning that it is the field of fractions of an image of the tensor product $F_1 \otimes_P F$).

Getting more specific, suppose $p_1 \neq p_2$ are primes. Let G be the infinite cyclic profinite p_2 -group, the inverse limit of all $\mathbb{Z}/p_2^n\mathbb{Z}$. Take F_1/\mathbb{Q} Galois with group G . (For example, F_1 could be contained in the infinite extension of \mathbb{Q} obtained by adjoining all p_2^n roots of 1.) Note that G has no finite subgroups. The field extension F_1F/F has Galois group a subgroup of G , and must be finite dimensional (being inside E/F), and so must be trivial. That is, $F_1F = F$, implying $F_1 \subseteq F$.

Next, take D'_1/\mathbb{Q} of degree p_1 and let $D_1 = D'_1 \otimes_{\mathbb{Q}} F_1$, a division algebra since F_1/\mathbb{Q} is a pro- p_2 extension, and let D_2/\mathbb{Q} be any division algebra split by F_1 . For example, there is a cyclic degree p_2 extension L/\mathbb{Q} such that $L \subset F_1$. By class field theory there is a degree p_2 division algebra with maximal subfield L .

Proposition 1.1. *There is no division algebra E/F containing both D_1 and D_2 .*

Proof. If E/F contained both D_1 and D_2 , then $F_1 \subseteq F$ is central, by the above paragraph, so E contains D_2F_1 which is a division algebra but also a homomorphic image of the split algebra $D_2 \otimes_{\mathbb{Q}} F_1$ and thus is commutative, a contradiction.

The rationale for this example is that the centers are incompatible in some sense.

2. POSITIVE RESULTS

Remark 2.1. *Since every division algebra is a tensor product of division algebras of prime power degree, it is natural to ask that if D_1/F_1 and D_2/F_2 are division algebras over respective degrees p^{t_1} and p^{t_2} , then can D_1/F_1 and D_2/F_2 be embedded into a single division algebra E/F of degree p^t , and is there a bound for t in terms of t_1 and t_2 ? What would be the best bound?*

We approach the problem via [Sa]. First let us fix some notation. We write $\text{index}(D)$ for the (Schur) index of D . Fixing $r > 1$, let $UD(F, n)/Z(F, n)$ denote the generic division algebra of degree n over F in r indeterminates. We write Z for $Z(F, n)$. When L/F is a cyclic Galois extension of dimension n and $a \in F$, $\Delta(L/F, a)/F$ denotes the F -central cyclic algebra having maximal subfield L , together with some element z inducing the automorphism generating $\text{Gal}(L/F)$, satisfying $z^n = a$. We begin with some lemmas.

Lemma 2.2. *There is a field $K(t) \supset Z$ and a degree n cyclic extension $L/K(t)$ such that L/F is rational, and $UD(F, n) \otimes_Z K(t) = \Delta(L/K(t), t)$.*

Proof. Write $Z = F(X \oplus Y)^{S_n}$ as usual (see [Sa] p. 322). Let $C_n \subset S_n$ be generated by the n cycle $(1, 2, \dots, n)$. Over C_n , $Y \cong M \oplus \mathbb{Z}$ where M is a free C_n lattice. We can set $L = F(X \oplus M)$, $K = L^{C_n}$, and take t to be the generator of \mathbb{Z} .

Lemma 2.3. *Suppose F is a field and D/F is a division algebra. Set*

$$A = D \otimes_F UD(F, n)^b = (D \otimes_F Z) \otimes_Z UD(F, n)^b.$$

Then $\text{index}(A)$ is the degree of D times $n/(n, b) = \text{index}(UD(F, n)^b)$.

Proof. Since $UD(F, n)$ has index equal exponent, the index of any power is equal to the exponent. In fact, if $b = b'(n, b)$ and $n = n'(n, b)$ then $(b', n') = 1$. Thus $UD(F, n)^b = (UD(F, n)^{(n, b)})^{b'}$ and the index and exponent of $UD(F, n)^b$ is the same as that of $(UD(F, n)^{(n, b)})$. That is, we may assume $b|n$.

By Lemma 2.2, there is a field $K(t) \supset Z$ and a degree n cyclic extension $L/K(t)$ such that $D \otimes_F L$ is a division algebra and $UD(F, n) \otimes_Z K(t) = \Delta(L/K(t), t)$. Of course, by Galois theory, $\Delta(L/K(t), t)^b$ is equal in the Brauer group to $\Delta(L'/K(t), t)$ where L/L' has degree b . Finally, $(D \otimes_F K(t)) \otimes_{K(t)} \Delta(L'/K(t), t)$ is a division algebra via twisted polynomial rings.

We are in the game of embedding division algebras into bigger division algebras. The key method is the following.

Theorem 2.4. *Suppose D/K is a division algebra of degree a and K/F has degree b . Assume E/F is a division algebra of degree $N = nab$. Then D is isomorphic to a subalgebra of E over F if and only if $(E \otimes_F K) \otimes_K D^\circ$ has (Schur) index dividing n . Furthermore, if this index divides n then it is equal to n .*

Proof. Suppose $D \subset E$. In particular, $K \subset E$ and so $E \otimes_F K$ has index N/b and we set E'/K to be the associated division algebra which is the centralizer of K in E . Then $D \subset E'$ and we take D' to be its centralizer, implying $E' = D \otimes_K D'$. Since D' has degree n , we have proven one direction.

Conversely, suppose $\text{index}((E \otimes_F K) \otimes_K D^{\circ p})$ divides n . Then $\text{index}(E \otimes_F K)$ divides na , implying $\text{index}(E \otimes_F K) = na$ since $\text{index}(E \otimes_F K) \geq \frac{N}{b} = na$. Thus,

$(E \otimes_F K) \otimes_K D^{\text{op}} \sim A$, where A/K has degree n , implying $[E \otimes_F K]$ is equal in the Brauer group to $[D][A]$.

Let E' be the centralizer of K in E . Since the degrees agree, $E' \cong D \otimes_K A$.

We are going to force one algebra inside another by using partial splitting fields and Weil transfers. More specifically, let A/K be a central simple algebra and n an integer dividing the degree of A/K . Let $V_n(A)$ be the variety of rank n left ideals of A and let $K_n(A)$ be its field of fractions. Then for any field $K' \supset K$, $V_n(A)$ has a K' point if and only if $A \otimes_K K'$ has index dividing n .

Next we set $W_n(A)$ to be the Weil transfer to F of $V_n(A)$, so for $F' \supset F$, $W_n(A)$ has an F' point if and only if $V_n(A)$ has an $K \otimes_F F'$ point (and in fact there is a natural correspondence). Let $F_n(A)$ denote the field of fractions of $W_n(A)$. Then $\text{index}(A \otimes_F K F_n(A))$ divides n .

The important tool for using this construction is the following result ([Sa]) about index reduction, for which we need to introduce more notation. Let K/F be finite separable with Galois closure \bar{K}/F . Let G be the Galois group of \bar{K}/F and $H \subset G$ the subgroup corresponding to \bar{K}/K . If r is the degree of A/K , then we can define an “action” of the G module $R = (\mathbb{Z}/r\mathbb{Z})[G/H]$ as follows. Let $\bar{A} = A \otimes_K \bar{K}$, so H has a natural semilinear action on \bar{A} and for any $g \in G$ we can define the g twist $g(\bar{A})$. Of course, for $g \in G$, gHg^{-1} has a natural semilinear action on $g(\bar{A})$.

For $\alpha \in R$, define $H_\alpha = \{g \in G | g\alpha = \alpha\}$. Define $K(\alpha) = \bar{K}^{H_\alpha}$. Write

$$\alpha = \sum n_{gH} gH;$$

then the n_{gH} are constant on H_α -orbits. Fix a coset gH and set $e = n_{gH}$. Let $L \subset H_\alpha$ be the stabilizer of gH . Let $\mathcal{O} = \{g_i H\}$ be the orbit of H_α containing gH so $e = n_{g_i H}$ for all i . Then L acts naturally on $g(\bar{A})$ and H_α acts on B_{gH} which is the tensor product over \bar{K} of $g_i(\bar{A})^e$, one for each $g_i H$ in \mathcal{O} .

Now we let gH vary, one for each H_α orbit. Tensor over \bar{K} all the B_{gH} defined above and call the resulting \bar{K} algebra B . Note that \bar{K} is the center of B . Define A^α to be the H_α invariant subring of B . Then A^α has center $K(\alpha)$.

Finally, for $\alpha = \sum n_{gH} gH$ as above, define

$$|\alpha| = \prod_{gH} \frac{n}{(n, n_{gH})}. \quad (1)$$

Theorem 2.5. ([Sa] p. 332). *Notation as above, suppose B/F is any central simple algebra (over F). Then the index of $B \otimes_F F_n(A)$ is the gcd of all the integers*

$$\text{index}(B \otimes_F A^\alpha)[K(\alpha) : F]|\alpha|,$$

taken over all $\alpha \in R$.

We actually need a double version of the above result. Let us assume that K/F and K'/F are finite separable with Galois closures \bar{K}/F and \bar{K}'/F and corresponding groups $G \supset H$ and $G' \supset H'$. For convenience we may assume that \bar{K}/F and \bar{K}'/F are linearly disjoint. Let A/K and A'/K' be central simple algebras and let $F_{n,n'}(A, A')$ denote the join of the fields $F_n(A)$ and $F_{n'}(A')$ over F . If A'/K' has degree r' set $R' = (\mathbb{Z}/r'\mathbb{Z})[G'/H']$ as above. If $\alpha \in R$ and $\beta \in R'$ set $K(\alpha, \beta) = K(\alpha) \otimes_F K'(\beta)$.

Finally write $\beta = \sum_{gH'} m_{gH'} gH' \in R'$ and set

$$|\beta| = \prod_{gH'} \frac{n'}{(n', m_{gH'})}.$$

Theorem 2.6. *Suppose B/F is a central simple algebra and set*

$$B(\alpha, \beta) = B \otimes_F K(\alpha, \beta).$$

Then the (Schur) index $\mathbf{i} := \text{index}(B \otimes_F F_{n,n'}(A, A'))$ is the gcd of the integers

$\text{index}((B(\alpha, \beta) \otimes_{K(\alpha, \beta)} (A^\alpha \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (A'^\beta \otimes_{K'(\beta)} K(\alpha, \beta))) [K(\alpha, \beta) : F] |\alpha| |\beta|,$
ranging over all $\alpha \in R$ and $\beta \in R'$.

Proof. The basic idea here is to apply Theorem 2.5 twice, noting that $[K(\alpha, \beta) : F] = [K(\alpha) : F][K'(\beta) : F]$. Put $B' = B \otimes_F F_n(A)$. Then, by Theorem 2.5, using the fact that $[F_n K'(\beta) : F_n(A)] = [K'(\beta) : F]$, \mathbf{i} is the gcd of all integers

$$\text{index}((B' \otimes_{F_n(A)} F_n K'(\beta)) \otimes_{F_n K'(\beta)} (A'^\beta \otimes_{K'(\beta)} F_n K'(\beta))) [K'(\beta) : F] |\beta|,$$

where $F_n K'(\beta)$ is the join of $F_n(A)$ and $K'(\beta)$ over F . Note that $F_n K'(\beta)$ is the function field of $W_n(A \otimes_F K(\beta))$ which is the $K(\alpha, \beta)/K'(\beta)$ transfer of $V_n(A \otimes_F K'(\beta))$. Now by Theorem 2.5 again, using B' instead of B , each

$$\text{index}((B' \otimes_{F_n(A)} F_n K'(\beta)) \otimes_{F_n K'(\beta)} (A'^\beta \otimes_{K'(\beta)} F_n K'(\beta)))$$

is the gcd of

$$\text{index}((B \otimes_F A^\alpha) \otimes_{K(\alpha)} (A'^\beta \otimes_{F_n K'(\beta)} K(\alpha, \beta))) [K(\alpha, \beta) : K'(\beta)] |\alpha|$$

and the result follows.

Now suppose we already know that D_1/K_1 and D_2/K_2 are division algebras of degrees d_i and K_i/F is separable of degree e_i . Set $m_i = d_i e_i$ and let N be any multiple of the lcm of m_1^2 and m_2^2 . Recall that F'/F is called a regular field extension if F' is finitely generated as a field over F , F' is a finite separable extension of a purely transcendental extension of F , and F is algebraically closed in F' .

Theorem 2.7. *There is a regular field extension $F' \supset F$ and a division algebra E'/F' of degree N such that $D_i \subset E'$ is compatible with $F \subset F'$ for $i = 1, 2$.*

Proof. Set $n_i = N/m_i$, noting that n_i is a multiple of m_i . Set

$$E = UD(F, N)$$

and we will extend Z so that the D_i embed in the base extension of E . To achieve this we set $F' = F_{n_1, n_2}(A_1, A_2)$ where

$$A_i = (D_i^\circ \otimes_{K_i} K_i Z) \otimes_{K_i Z} (E \otimes_Z K_i Z).$$

Note that $(E \otimes_Z K_i Z)$ is just the generic division algebra over K_i . Also note that $E' = E \otimes_Z F'$ has both D_i embedded because we have suitably reduced the index of both A_i . The problem is to show that E' is a division algebra, i.e., that $\text{index}(E') = N$, and for this we apply Theorem 2.6. In applying this theorem, note that the degree of A_1 is $N d_1$, and so is a multiple of n_1 . We make a similar comment about the degree of A_2 .

To apply Theorem 2.6, we need to get a handle on

$$\text{index}((E \otimes_Z ZK(\alpha, \beta))^{1+\alpha+\beta} \otimes (D_1^{-\alpha} \otimes_{K(\alpha)} ZK(\alpha, \beta)) \otimes (D_2^{-\beta} \otimes_{K'(\beta)} ZK(\alpha, \beta))),$$

where the unsubscripted tensors are over $ZK(\alpha, \beta)$. Write $\alpha = \sum n_{gH}gH$ and set $a = \sum n_{gH}$ and similarly for β and b . Note that E is not moved by either Galois group so $(E \otimes_Z K(\alpha, \beta)Z)^{1+\alpha+\beta}$ is $E^{1+a+b} \otimes_Z K(\alpha, \beta)Z$ which has index $N/(N, 1+a+b)$ which we define to be $N_{a,b}$. By Lemma 2.3 the above index is

$$\text{index}((D_1^{-\alpha} \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (D_2^{-\beta} \otimes_{K'(\beta)} K(\alpha, \beta))N_{a,b}$$

and so we want to show that N divides the expression

$$\text{index}((D_1^\alpha \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (D_2^\beta \otimes_{K'(\beta)} K(\alpha, \beta))[K(\alpha, \beta) : F]|\alpha||\beta|N_{a,b}. \quad (2)$$

We show the needed divisibility prime by prime. So assume, for p prime that p^s divides N exactly, in the sense that $\frac{N}{p^s}$ is prime to p . Likewise, assume that p^{t_i} divides n_i exactly. Since $N = n_i d_i e_i$ and n_i is a multiple of $d_i e_i$ we have $2t_i \geq s$ and also $t_1 + t_2 \geq s$. If $1+a+b$ is prime to p we are done. Thus we assume p divides

$$1 + \sum_{gH} n_{gH} + \sum_{gH'} m_{gH'} \quad (3)$$

and this implies that at least one summand in

$$\sum_{gH} n_{gh} + \sum_{gH'} m_{gH'} \quad (4)$$

is prime to p . If any term in (4), say n_{gH} , is prime to p , then $\frac{n}{(n, n_{gH})}$ is divisible by p^{t_1} . Thus, if two terms in (4) are prime to p , then p^{2t_1} or $p^{t_1+t_2}$ or p^{2t_2} divide $|\alpha||\beta|$ and again we are done.

Thus we assume that p is prime to exactly one summand in (4). Replacing α by $g^{-1}\alpha$, we assume that only n_H is prime to p . It follows that H_α fixes the trivial coset H and so $H_\alpha \subseteq H$, implying $K(\alpha) \supseteq K$. Set $\alpha' = \alpha - n_H H$; thus,

$$|\alpha'| = \prod_{gH \neq H} \frac{n}{(n, n_{gH})}.$$

We know that $K(\alpha) = [K(\alpha) : K][K : F]$. Write $s = s_1 + s_2 + s_3$ where p^{s_1} is the exact power of p dividing n , p^{s_2} is the exact power dividing d_1 , and p^{s_3} is the exact power dividing e_1 . Note that p^{s_1} divides $\frac{n}{(n, n_H)}$, and of course p^{s_3} divides $[K : F]$. Thus it suffices to show that p^{s_2} divides

$$\text{index}(D'')[K(\alpha, \beta) : K]|\alpha'||\beta|, \quad (5)$$

where

$$D'' = (D_1^\alpha \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (D_2^\beta \otimes_{K'(\beta)} K(\alpha, \beta)).$$

We will prove in fact that (5) is divisible by d_1 . Note that

$$(D_1^\alpha \otimes_{K(\alpha)} K(\alpha, \beta)) = (D_1 \otimes_K K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (D_1^{\alpha'} \otimes_{K(\alpha)} K(\alpha, \beta)).$$

We need to estimate some indices. Of course D_1 has index d_1 , and so over $\bar{K}\bar{K}'$, $g(D_1)^{n_{gH}}$ has index dividing $\frac{d_1}{(d_1, n_{gH})}$. Then $D_1^{\alpha'} \otimes_{K(\alpha)} K(\alpha, \beta)$ has index dividing $\prod_{gH \neq H} \frac{d_1}{(d_1, n_{gH})}$. Similarly $D_2^\beta \otimes_{K'(\beta)} K(\alpha, \beta)$ has index dividing $\prod_{gH'} \frac{d_2}{(d_2, m_{gH'})}$.

We need a trivial lemma.

Lemma 2.8. *Suppose a divides b . Then $a/(a, d)$ divides $b/(b, d)$.*

Proof. For any prime p , the power of p dividing $\frac{a}{(a, d)}$ is less than or equal to the power of p dividing $\frac{b}{(b, d)}$.

Let p^{u_1} be the exact power of p dividing $\text{index}(D_1 \otimes_K K(\alpha, \beta))$ and let

$$D^\# = (D_1^{\alpha'} \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (D_2^\beta \otimes_{K'(\beta)} K(\alpha, \beta)).$$

Also let p^{u_2} be the exact power of p dividing $\text{index}(D^\#)$. It follows from Lemma 2.8 that p^{u_2} divides $|\alpha'| |\beta|$. Also, $D'' = (D_1^{n_H} \otimes_K K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} D^\#$. Now let p^{u_3} be the exact power of p dividing

$$\text{index}(D_1^{n_H} \otimes_K K(\alpha, \beta)).$$

This is the same as the exact power of p dividing

$$\text{index}(D_1 \otimes_K K(\alpha, \beta)).$$

Set p^{u_4} to be the exact power of p dividing $[K(\alpha, \beta) : K]$. Then $\text{index}(D'')$ is a multiple of $p^{u_1 - u_2}$. Thus $\text{index}(D'') |\alpha'| |\beta|$ is a multiple of p^{u_1} and

$$\text{index}(D'') |\alpha'| |\beta| [K(\alpha, \beta) : K]$$

is a multiple of $p^{u_1 + u_4}$ which is the exact power of p dividing

$$\text{index}(D_1 \otimes_K K(\alpha, \beta)) [K(\alpha, \beta) : K],$$

a multiple of d_1 . This proves Theorem 2.7.

Finally let D'_i/K'_i be arbitrary such that both K_i are finitely generated over a prime or algebraically closed field k' . Let k be the subfield of elements of K_1 and K_2 algebraic over k' . Then k/k' is a finite field extension. In particular, k is perfect. Then we can write $K'_i \supset F_i$ such that K_i/F_i is finite separable and F_i/k is rational. Let $F = q(F_1 \otimes_k F_2)$, $K_i = K'_i \otimes_{F_i} F$ and $D_i = D'_i \otimes_{K'_i} K_i$. We apply Theorem 2.7 to the D_i and conclude:

Theorem 2.9. *Suppose D'_i/K'_i are division algebras with the K'_i finitely generated over a common field k' which is either algebraically closed or the prime field. Then the D'_i can be embedded into a common division algebra E finite over its center.*

The question remains: What is the lowest possible bound for $\deg E$?

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